

# On the process of the eigenvalues of a Hermitian Lévy process

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*Dedicated to Ole E. Barndorff-Nielsen on the occasion of his 80th Birthday*

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## Abstract

The dynamics of the eigenvalues (semimartingales) of a Lévy process  $X$  with values in Hermitian matrices is described in terms of Itô stochastic differential equations with jumps. This generalizes the well known Dyson–Brownian motion. The simultaneity of the jumps of the eigenvalues of  $X$  is also studied. If  $X$  has a jump at time  $t$  two different situations are considered, depending on the commutativity of  $X(t)$  and  $X(t-)$ . In the commutative case all the eigenvalues jump at time  $t$  only when the jump of  $X$  is of full rank. In the noncommutative case,  $X$  jumps at time  $t$  if and only if all the eigenvalues jump at that time when the jump of  $X$  is of rank one.

**Key words:** Dyson–Brownian motion, infinitely divisible random matrix, Bercovici–Pata bijection, matrix semimartingale, simultaneous jumps, non-colliding process, rank one perturbation, stochastic differential equation with jumps.

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# 1 Introduction

This paper is on a topic that meets a number of subjects where Ole Barndorff-Nielsen has contributed. One is Ole's interests in the study of infinitely divisible random matrix models and their associated matrix-valued processes, like subordinators, Lévy and Ornstein–Uhlenbeck processes [4]–[5], [8]–[10]. Another is the connection between classical and free infinite divisibility, where Ole's papers with Thorbjørnsen [11]–[12] drove the study of the so-called Upsilon transformations of classical infinitely divisible distributions [3], [6], [7].

Matrix Lévy processes with jumps of rank one arise as a covariation process of two  $\mathbb{C}^d$ –valued Lévy processes [23] and they provide natural examples of matrix subordinators and matrix Lévy processes of bounded variation, such as those considered in [4] and [8].

In the context of free probability, Bercovici and Pata [14] introduced a bijection from the set of classical infinitely divisible distributions on  $\mathbb{R}$  to the set of free infinitely divisible distributions on  $\mathbb{R}$ . This bijection was described in terms of ensembles of Hermitian random matrices  $\{X_d : d \geq 1\}$  by Benaych-Georges [13] and Cabanal-Duvillard [20]. For fixed  $d \geq 1$ , the associated  $d \times d$  matrix distribution of  $X_d$  is invariant under unitary conjugations and it is (matrix) infinitely divisible. The associated Hermitian Lévy process  $\{X_d(t) : t \geq 0\}$  has been considered in [23] and it has the property that, in the non pure Gaussian case, the jumps  $\Delta X_d(t) = X_d(t) - X_d(t-)$  are  $d \times d$  matrices of rank one, see also [22], [29].

The simplest example of this connection is the well-known theorem of Wigner that gives the semi-circle distribution (free infinitely divisible) as the asymptotic spectral distribution of  $\{X_d : d \geq 1\}$  in the case of the Gaussian Unitary Ensemble (GUE), see [1]. In this case, for each fixed  $d \geq 1$ , the associated Lévy process of  $X_d$  is the  $d \times d$  Hermitian Brownian motion  $\{B(t) : t \geq 0\} = \{(B_{jk}(t)) : t \geq 0\}$  where  $(B_{jj}(t))_{j=1}^d, (\operatorname{Re} B_{jk}(t))_{j < k}, (\operatorname{Im} B_{jk}(t))_{j < k}$  is a set of  $d^2$  independent one-dimensional Brownian motions with parameter  $\frac{t}{2}(1 + \delta_{jk})$ .

Still in the GUE case, let  $\lambda = \{(\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t)) : t \geq 0\}$  be the  $d$ -dimensional stochastic process of eigenvalues of  $B$ . In a pioneering work, Dyson [24] showed that if the eigenvalues start at different positions  $(\lambda_1(0) > \lambda_2(0) > \dots > \lambda_d(0)$  a.s.), then they never meet at any time  $(\lambda_1(t) > \lambda_2(t) > \dots > \lambda_d(t) \quad \forall t > 0$  a.s.) and furthermore they form a diffusion process (and a semimartingale) satisfying the Itô Stochastic Differential Equation (SDE)

$$d\lambda_i(t) = dW_i(t) + \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad t \geq 0, 1 \leq i \leq d, \quad (1)$$

where  $W_1, \dots, W_d$  are independent one-dimensional standard Brownian motions (see [1, Sec. 4.3.1]). The stochastic process  $\{\lambda(t) : t \geq 0\}$  is called the Dyson non-colliding Brownian motion corresponding to the GUE, or, in short, Dyson-Brownian motion. The study of the eigenvalue process of other continuous matrix-valued processes has received considerable attention, see [17]–[19], [25]–[28]. The eigenvalues of all these processes are strongly dependent and do not collide at any time almost surely.

The aim of this paper is to understand the behavior on time of the eigenvalue process  $\lambda = \{(\lambda_1(t), \lambda_2(t), \dots, \lambda_d(t)) : t \geq 0\}$  of a  $d \times d$  Hermitian Lévy process  $X = \{X(t) : t \geq 0\}$ . The first goal is to give conditions for the simultaneity of the jumps of the eigenvalues. In particular, when the jump of  $X$  is of rank one and  $X(t)$  and  $X(t-)$  do not commute, we show that a single eigenvalue jumps at time  $t$  if and only if all the eigenvalues jump at that time. This fact which is due to the strong dependence between the eigenvalue processes contrasts with the case of the no simultaneity of the jumps at any time of  $d$  independent real Lévy processes; in Example 6 we give a Hermitian Lévy process with such eigenvalue process. We observe that in general, the eigenvalue process is not a Lévy process (as in the Dyson-Brownian case) but a semimartingale given in terms of the Hermitian Lévy process  $X$ .

The second goal is to describe the dynamics of the eigenvalue process, analogously to (1). This leads to first considering the appropriate Itô's formula for matrix-valued semimartingales with jumps, more general than those of bounded variation considered by Barndorff-Nielsen and Stelzer [8].

Our main results and the structure of the paper are as follows. In Section 2 we gather several results on Hermitian Lévy processes, including versions of Itô's formula which are useful for studying the corresponding

eigenvalue processes. We also point out conditions under which the spectrum of  $X$  will be simple for every  $t > 0$  and conditions for the differentiability of the eigenvalues. In Section 3 we consider the simultaneous jumps of the eigenvalue (semimartingale) process of a  $d \times d$  Hermitian Lévy process  $X$ . When  $\Delta X(t) = X(t) - X(t-)$  is not zero and  $X(t)$  and  $X(t-)$  commute, we show that the eigenvalues jump simultaneously if and only if the matrix jump  $\Delta X(t)$  is of rank  $d$ . On the other hand, if  $\Delta X(t) \neq 0$  is of rank one and  $X(t)$  and  $X(t-)$  do not commute, then the process  $X$  jumps at time  $t > 0$  if and only if all the components of  $\lambda$  jump at that time. This leads to analyzing the relation between the eigenvalues of  $X(t)$  and  $X(t-)$  when the jump  $\Delta X(t)$  is a rank one matrix. A key step in this direction is the recent paper [31] on the spectrum of a rank one perturbation of an unstructured matrix. Extensions to Hermitian semimartingales are also pointed out. Finally, Section 4 considers the dynamics of the eigenvalues of a Hermitian Lévy process, extension to the Dyson-Brownian motion, showing that a repulsion force appears in the bounded variation part of the eigenvalues only when there is a Gaussian component of  $X$ . One of the main problems to overcome is the fact that even when  $X(t)$  is Hermitian and has a simple spectrum in an open subset almost surely,  $X(t-)$  may not.

We shall use either  $X(t)$  or  $X_t$  to denote a stochastic process in a convenient way when the dimension or the entries of a matrix have also to be specified.

## 2 Preliminaries on Hermitian Lévy processes

In this section we gather several results on matrix-valued semimartingales and general Hermitian Lévy processes.

Let  $\mathbb{M}_d = \mathbb{M}_d(\mathbb{C})$  denote the linear space of  $d \times d$  matrices with complex entries with scalar product  $\langle A, B \rangle = \text{tr}(B^* A)$  and the Frobenius norm  $\|A\| = [\text{tr}(A^* A)]^{1/2}$  where  $\text{tr}$  denotes the (non-normalized) trace. The set of Hermitian random matrices in  $\mathbb{M}_d$  is denoted by  $\mathbb{H}_d$ ,  $\mathbb{H}_d^0 = \mathbb{H}_d \setminus \{0\}$  and  $\mathbb{H}_d^1$  is the set of rank one matrices in  $\mathbb{H}_d$ .

We will use the identification of a  $d \times d$  Hermitian matrix  $X = (x_{ij} + iy_{ij})_{1 \leq i, j \leq d}$  with an element in  $\mathbb{R}^{d^2}$ , that is,

$$X = (x_{ij} + iy_{ij})_{1 \leq i, j \leq d} \leftrightarrow \widehat{X} := \left( x_{11}, \dots, x_{dd}, (x_{ij}, y_{ij})_{1 \leq i < j \leq d} \right)^\top. \quad (2)$$

Thus, the set  $\mathbb{H}_d$  is identified with  $\mathbb{R}^{d^2}$  and one has that for  $X, Y \in \mathbb{H}_d$ ,  $\text{tr}(XY) = \widehat{Y}^\top \widehat{X}$  is real.

An  $\mathbb{M}_d$ -valued process  $X = \{(x_{ij})(t) : t \geq 0\}$  is a matrix semimartingale if  $x_{ij}(t)$  is a complex semimartingale for each  $i, j = 1, \dots, d$ . Let  $X = \{(x_{ij})(t) : t \geq 0\} \in \mathbb{H}_d$  and  $Y = \{(y_{ij})(t) : t \geq 0\} \in \mathbb{H}_d$  be semimartingales. Similar to the case of matrices with real entries in [8], the matrix covariation of  $X$  and  $Y$  is defined as the  $\mathbb{H}_d$ -valued process  $[X, Y] := \{[X, Y](t) : t \geq 0\}$  with entries

$$[X, Y]_{ij}(t) = \sum_{k=1}^d [x_{ik}, y_{kj}](t), \quad (3)$$

where  $[x_{ik}, y_{kj}](t)$  is the covariation of the  $\mathbb{C}$ -valued semimartingales  $\{x_{ik}(t) : t \geq 0\}$  and  $\{y_{kj}(t) : t \geq 0\}$ , see [30, pp 83]. One has a decomposition into a continuous part and a pure jump part by

$$[X, Y](t) = [X^c, Y^c](t) + \sum_{s \leq t} \Delta X_s \Delta Y_s^*, \quad (4)$$

where  $[X^c, Y^c]_{ij}(t) := \sum_{k=1}^d [x_{ik}^c, y_{kj}^c](t)$ . We recall that for any semimartingale  $x$ , the process  $x^c$  is the a.s. unique continuous local martingale  $m$  such that  $[x - m]$  is purely discontinuous.

Matrix-valued Lévy processes are examples of matrix-valued semimartingales. In particular, so are the Hermitian Lévy processes.

## 2.1 The Lévy–Khintchine representation and Lévy–Itô decomposition

A  $d \times d$  matrix Hermitian Lévy process  $X = \{X(t) : t \geq 0\}$  is characterized by the Lévy–Khintchine representation of its Fourier transform  $\mathbb{E}e^{i\text{tr}(\Theta X(t))} = \exp(t\psi(\Theta))$  with Laplace exponent

$$\psi(\Theta) = i\text{tr}(\Theta\Psi) - \frac{1}{2}\text{tr}(\Theta\mathcal{A}\Theta) + \int_{\mathbb{H}_d} \left( e^{i\text{tr}(\Theta\xi)} - 1 - i\frac{\text{tr}(\Theta\xi)}{1 + \|\xi\|^2} \right) \nu(d\xi), \quad \Theta \in \mathbb{H}_d,$$

where  $\mathcal{A} : \mathbb{H}_d \rightarrow \mathbb{H}_d$  is a positive symmetric linear operator (i.e.,  $\text{tr}(\Phi\mathcal{A}\Phi) \geq 0$  for  $\Phi \in \mathbb{H}_d$  and  $\text{tr}(\Theta_2\mathcal{A}\Theta_1) = \text{tr}(\Theta_1\mathcal{A}\Theta_2)$  for  $\Theta_1, \Theta_2 \in \mathbb{H}_d$ ),  $\nu$  is a measure on  $\mathbb{H}_d$  (the Lévy measure) satisfying  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{H}_d} (1 \wedge \|x\|^2) \nu(dx) < \infty$ , and  $\Psi \in \mathbb{H}_d$ . The triplet  $(\mathcal{A}, \nu, \Psi)$  uniquely determines the distribution of  $X$ .

Interesting matrix Gaussian distributions for some examples of  $\mathcal{A}$  are the following (here  $\Psi = 0$  and  $\nu = 0$ ):

- (i)  $\mathcal{A} = \sigma^2 I_d$ ,  $\sigma^2 > 0$ , corresponds to the GUE case of parameter  $\sigma^2$ ;
- (ii) More generally,  $\mathcal{A}\Theta = \Sigma_1\Theta\Sigma_2$ , for  $\Sigma_1, \Sigma_2$  nonnegative definite matrices in  $\mathbb{H}_d$ , corresponds to the matrix Gaussian distribution with Kronecker covariance  $\Sigma_1 \otimes \Sigma_2$ ;
- (iii)  $\mathcal{A}\Theta = \text{tr}(\Theta)\sigma^2 I_d$ ,  $\sigma^2 > 0$ , is the covariance operator of the Gaussian random matrix  $gI_d$  where  $g$  is a one-dimensional random variable with zero mean and variance  $\sigma^2$ .

Any  $\mathbb{H}_d$ -valued Lévy process  $X = \{X(t) : t \geq 0\}$  with triplet  $(\mathcal{A}, \nu, \Psi)$  is a semimartingale with the Lévy–Itô decomposition

$$X(t) = t\Psi + B_{\mathcal{A}}(t) + \int_{[0,t]} \int_{\|y\| \leq 1} y \tilde{J}_X(ds, dy) + \int_{[0,t]} \int_{\|y\| > 1} y J_X(ds, dy), \quad t \geq 0, \quad (5)$$

where  $\{B_{\mathcal{A}}(t) : t \geq 0\}$  is a  $\mathbb{H}_d$ -valued Brownian motion with covariance  $\mathcal{A}$ , i.e., it is a Lévy process with continuous sample paths (a.s.) and each  $B_{\mathcal{A}}(t)$  is centered Gaussian with

$$\mathbb{E} \{ \text{tr}(\Theta_1 B_{\mathcal{A}}(s)) \text{tr}(\Theta_2 B_{\mathcal{A}}(t)) \} = \min(s, t) \text{tr}(\Theta_1 \mathcal{A} \Theta_2) \text{ for each } \Theta_1, \Theta_2 \in \mathbb{H}_d;$$

$J_X(\cdot, \cdot)$  is a Poisson random measure of jumps on  $[0, \infty) \times \mathbb{H}_d^0$  with intensity measure  $\text{Leb} \otimes \nu$ , independent of  $\{B_{\mathcal{A}}(t) : t \geq 0\}$  and the compensated measure is given by

$$\tilde{J}_X(dt, dy) = J_X(dt, dy) - dt\nu(dy).$$

For a systematic study of Lévy processes with values in Banach spaces, see [2].

## 2.2 Hermitian Lévy processes with simple spectrum

Hermitian Lévy processes with simple spectrum (all eigenvalues distinct) are of special importance in this paper. For each  $t > 0$ , if  $X(t)$  has an absolutely continuous distribution then  $X(t)$  has a simple spectrum almost surely, since the set of Hermitian matrices whose spectrum is not simple has zero Lebesgue measure [34], see also [1]. The absolute continuity gives also that  $X(t)$  is a nonsingular random matrix since for a Hermitian matrix  $A$ ,  $\det(A)$  is a real polynomial in  $d^2$  variables and  $\{A \in \mathbb{H}_d : \det(A) = 0\}$  has zero Lebesgue measure in  $\mathbb{R}^{d^2}$ .

If there is a Gaussian component, i.e.,  $\mathcal{A} \neq 0$ , then  $X(t)$  has an absolutely continuous distribution for each  $t > 0$ . Next we point out a sufficient condition on the Lévy measure of a Hermitian Lévy process without Gaussian component to have an absolutely continuous distribution and hence a simple spectrum for each fixed time.

Let  $\mathbb{S}_d$  be the unit sphere of  $\mathbb{H}_d$ . A polar decomposition of the Lévy measure  $\nu$  is a family of pairs  $(\pi, \rho_\xi)$  where  $\pi$ , the spherical component, is a finite measure on  $\mathbb{S}_d$  with  $\pi(\mathbb{S}_d) \geq 0$  and  $\rho_\xi$ , the radial component, is a measure on  $(0, \infty)$  for each  $\xi \in \mathbb{S}_d$  with  $\rho_\xi((0, \infty)) > 0$  such that

$$\nu(E) = \int_{\mathbb{S}_d} \pi(d\xi) \int_{(0, \infty)} 1_E(u\xi) \rho_\xi(du), \quad E \in \mathcal{B}(\mathbb{H}_d^0).$$

We say that  $\nu$  satisfies the condition **D** if, for each  $\xi \in \mathbb{S}_d$ , there is a nonnegative function  $g_\xi$  on  $(0, \infty)$  such that  $\rho_\xi(B) = \int_B g_\xi(u) du$  for any  $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$  and the following divergence condition holds

$$\int_{(0, \infty)} g_\xi(u) du = \infty \quad \pi\text{-a.e. } \xi.$$

As an example, recall that a random matrix  $M$  is called self-decomposable (and hence infinitely divisible) if for any  $b \in (0, 1)$  there exists  $M_b$  independent of  $M$  such that  $M$  and  $bM + M_b$  have the same matrix distribution. An infinitely divisible random matrix  $M$  in  $\mathbb{H}_d$  with triplet  $(\mathcal{A}, \nu, \Psi)$  is self-decomposable if and only if

$$\nu(E) = \int_{\mathbb{S}_d} \pi(d\xi) \int_{(0, \infty)} 1_E(u\xi) \frac{k_\xi(u)}{u} (du), \quad E \in \mathcal{B}(\mathbb{H}_d^0),$$

with a finite measure  $\pi$  on  $\mathbb{S}_d$  and a nonnegative decreasing function  $k_\xi$  on  $(0, \infty)$  for each  $\xi \in \mathbb{S}_d$ . In this case  $\nu$  satisfies the condition **D** for  $g_\xi(u) = k_\xi(u)/u$  where we can choose the measure  $\pi$  to be zero on the set  $\{\xi \in \mathbb{S}_d : k_\xi(0+) = 0\}$ .

The following proposition for Hermitian Lévy processes follows from the corresponding real vector case, Theorems 27.10 and 27.13 in [32], by using the identification (2) of  $\mathbb{H}_d$  with  $\mathbb{R}^{d^2}$ . We recall that a Lévy process is nondegenerate if its distribution for any fixed time is nondegenerate.

**Proposition 1** (a) *If  $\{X(t) : t \geq 0\}$  is a nondegenerate Lévy process in  $\mathbb{H}_d$  without Gaussian component satisfying condition **D**, then  $X(t)$  has absolutely continuous distribution for each  $t > 0$ .*

(b) *If  $\{X(t) : t \geq 0\}$  is a nondegenerate self-decomposable Lévy process in  $\mathbb{H}_d$  without Gaussian component, then  $X(t)$  has absolutely continuous distribution for each  $t > 0$ .*

## 2.3 Smoothness of the spectrum of Hermitian matrices

Next we consider several facts about eigenvalues as functions of the entries of a Hermitian matrix, similar to [28] for symmetric matrices, that will be useful in the sequel.

Following [1], let  $\mathbb{U}_d^{VG}$  be the set of unitary matrices  $U = (u_{ij})$ , with  $u_{ii} > 0$  for all  $i$ ,  $u_{ij} \neq 0$  for  $i, j$ , and all minors of  $U$  having non-zero determinants. Let us denote by  $\mathbb{H}_d^{VG}$  the set of matrices  $X$  in  $\mathbb{H}_d$  such that

$$X = UDU^*, \tag{6}$$

where  $D$  is a diagonal matrix with entries  $\lambda_i = D_{ii}$  such that  $\lambda_1 > \lambda_2 > \dots > \lambda_d$ ,  $U \in \mathbb{U}_d^{VG}$ . An element of  $\mathbb{H}_d^{VG}$  is called a very good matrix. The set of very good matrices  $\mathbb{H}_d^{VG}$  can be identified with an open subset of  $\mathbb{R}^{d^2}$ . It is known that the complement of  $\mathbb{H}_d^{VG}$  has zero Lebesgue measure. If  $T : \mathbb{U}_d^{VG} \rightarrow \mathbb{R}^{d(d-1)}$  is the mapping defined by

$$T(U) = \left( \frac{u_{12}}{u_{11}}, \dots, \frac{u_{1d}}{u_{11}}, \frac{u_{23}}{u_{22}}, \dots, \frac{u_{2d}}{u_{22}}, \dots, \frac{u_{(d-1)d}}{u_{(d-1)(d-1)}} \right),$$

then Lemma 2.5.6 in [1] shows that  $T$  is bijective, smooth, and the complement of  $\mathbb{U}_d^{VG}$  is a closed subset of zero Lebesgue measure.

Let  $\mathcal{S}_d$  be the open set

$$\mathcal{S}_d = \{(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{R}^d : \lambda_1 > \lambda_2 > \dots > \lambda_d\} \tag{7}$$

and let  $D_\lambda$  be the diagonal matrix  $D_{ii} = \lambda_i$  for any  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathcal{S}_d$ . Then the map  $\tilde{T} : \mathcal{S}_d \times T(\mathbb{U}_d^{VG}) \rightarrow \mathbb{H}_d^{VG}$  defined by  $\tilde{T}(\lambda, z) = T^{-1}(z)^* D_\lambda T^{-1}(z)$  is a smooth bijection and the inverse mapping of  $\tilde{T}$  satisfies

$$\tilde{T}^{-1}(X) = \left( \lambda(X), \tilde{U}(X) \right),$$

where  $\tilde{U}(X) = T(U)$  for  $U \in \mathbb{U}_d^{VG}$  satisfying (6). As a consequence of these results, there exist an open subset  $\tilde{G} \subset \mathbb{R}^{d^2}$  such that  $\mathbb{R}^{d^2} \setminus \tilde{G}$  has zero Lebesgue measure and a smooth function

$$\Phi : \mathbb{R}^{d^2} \rightarrow \mathbb{R}^d \quad (8)$$

such that  $\lambda(X) = \Phi(X)$  for  $X \in \tilde{G}$ .

## 2.4 The Itô Formula for Hermitian Lévy processes

We shall first consider a convenient notation for the Fréchet derivative of a function  $f$  from  $\mathbb{H}_d$  to  $\mathbb{R}$  (see [15, X.4]), using the identification (2) of an element  $X = (x_{ij} + iy_{ij})_{1 \leq i, j \leq d}$  in  $\mathbb{H}_d$  as the element  $(x_{11}, \dots, x_{dd}, (x_{ij}, y_{ij})_{1 \leq i < j \leq d})^\top$  in  $\mathbb{R}^{d^2}$ .

The derivative of  $f$  at the point  $u$  is denoted by  $Df(u)$ ; it can be thought as a linear operator from  $\mathbb{R}^{d^2}$  to  $\mathbb{R}$ . Its representation as a Hermitian matrix is

$$Df = \left( \frac{\partial}{\partial x_{ij}} f + i \frac{\partial}{\partial y_{ij}} f 1_{\{i \neq j\}} \right)_{1 \leq i \leq j \leq d}. \quad (9)$$

Let  $L(\mathbb{R}^{d^2}, \mathbb{R})$  denote the space of linear operators from  $\mathbb{R}^{d^2}$  to  $\mathbb{R}$ . If the map  $Df : \mathbb{R}^{d^2} \rightarrow L(\mathbb{R}^{d^2}, \mathbb{R})$  defined by  $u \rightarrow Df(u)$  is differentiable at a point  $u$ , we say that  $f$  is twice differentiable at  $u$ . We denote the derivative of the map  $Df$  at the point  $u$  by  $D^2f(u)$ . Recall that  $D^2f(u)$  is a bilinear operator from  $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$  to  $\mathbb{R}$  which is symmetric, that is,  $D^2f(u)(x, y) = D^2f(u)(y, x)$ . We use the following notation for the first and second partial derivative operators, where  $z_{ij} = x_{ij}$  or  $z_{ij} = y_{ij}$ , with  $y_{ii} = 0$ ,  $i = 1, \dots, d$

$$\sum_{(i,j)} \frac{\partial}{\partial z_{ij}} := \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_{ii}} + 2 \sum_{1 \leq i < j \leq d} \left( \frac{\partial}{\partial x_{ij}} + \frac{\partial}{\partial y_{ij}} \right), \quad (10)$$

$$\begin{aligned} \sum_{(i,j)(k,h)} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} &:= \sum_{1 \leq i \leq j \leq d} \frac{\partial^2}{\partial x_{ii} \partial x_{jj}} + \sum_{1 \leq i \leq j \leq d} \sum_{1 \leq k < h \leq d} \left( \frac{\partial^2}{\partial x_{ij} \partial x_{kh}} + \frac{\partial^2}{\partial x_{ij} \partial y_{kh}} \right) \\ &+ \sum_{1 \leq i < j \leq d} \sum_{1 \leq k < h \leq d} \frac{\partial^2}{\partial y_{ij} \partial y_{kh}}. \end{aligned} \quad (11)$$

For a linear operator  $\mathcal{A} : \mathbb{H}_d \rightarrow \mathbb{H}_d$ , we denote its  $d^2 \times d^2$  matrix representation by  $\mathcal{A} = (A_{ij, kh})$ .

Following Barndorff-Nielsen and Stelzer [8], we say that for each open subset  $G \subset \mathbb{M}_d$ , the process  $X = \{X_t : t \geq 0\}$  is locally bounded within  $G$  if there exist a sequence of stopping times  $(T_n)_{n \geq 1}$  increasing to infinity almost surely and a sequence of compact convex subsets  $D_n \subset G$  with  $D_n \subset D_{n+1}$  such that  $X_t \in D_n$  for all  $0 \leq t < T_n$ . Observe that if the process  $X$  is locally bounded within  $G$ , it cannot get arbitrarily close to the boundary of  $G$  in finite time, hence  $X_t$  and  $X_{t-}$  are in  $D_n$  at all times  $t \geq 0$ .

The following result can be proved similarly to Proposition 3.10.10 in Bichteler [16], who considered the multivariate case. The bounded variation multivariate case can be seen in [8].

**Theorem 2 (The Itô formula for matrix semimartingales)** *Let  $\{X_t : t \geq 0\}$  be a matrix semimartingale,  $G \subset \mathbb{H}_d$  an open subset, and  $f : G \rightarrow \mathbb{R}$  continuously differentiable. If  $\{X_t : t \geq 0\}$  is locally bounded within  $G$ , then  $X_{t-}$  belongs to  $G$  for all  $t > 0$ , the process  $f(X_t)$  is a semimartingale, and*

$$\begin{aligned} f(X_t) &= f(X_0) + \text{tr} \left( \int_{0+}^t Df(X_{s-}) dX_s \right) + \frac{1}{2} \int_{0+}^t \sum_{(i,j)(k,h)} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} f(X_s) d[z_{ij}^c, z_{kh}^c]_s \\ &+ \int_{(0,t] \times \mathbb{H}_d^0} [f(X_s) - f(X_{s-}) - \text{tr}(Df(X_{s-})y)] J_X(ds, dy). \end{aligned}$$

The following two propositions for Hermitian Lévy processes are special cases of the above result. The first one is the analog of the multivariate case in [21, Prop. 8.15].

**Proposition 3 (The Itô formula for matrix Lévy processes)** *Let  $\{X_t : t \geq 0\}$  be a Hermitian Lévy process with triplet  $(\mathcal{A}, \nu, \Psi)$ ,  $G \subset \mathbb{H}_d$  an open subset, and  $f : G \rightarrow \mathbb{R}$  twice continuously differentiable. If  $\{X_t : t \geq 0\}$  is locally bounded within  $G$ , then  $X_{t-}$  belongs to  $G$  for all  $t > 0$ , the process  $f(X_t)$  is a semimartingale, and*

$$\begin{aligned} f(X_t) = f(X_0) + \operatorname{tr} \left( \int_0^t Df(X_{s-}) dX_s \right) + \frac{1}{2} \int_0^t \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} f(X_s) ds \\ + \int_{(0,t] \times \mathbb{H}_d^0} [f(X_{s-} + y) - f(X_{s-}) - \operatorname{tr}(Df(X_{s-})y)] J_X(ds, dy). \end{aligned} \quad (12)$$

The next martingale-drift decomposition of functions of a Hermitian Lévy process  $X$  can be obtained by replacing the Lévy-Itô decomposition (5) for  $X$  into the stochastic integral part of (12).

**Proposition 4** *Let  $\{X_t : t \geq 0\}$  be a Hermitian Lévy process with triplet  $(\mathcal{A}, \nu, \Psi)$ ,  $G \subset \mathbb{H}_d$  an open subset, and  $f : G \rightarrow \mathbb{R}$  twice continuously differentiable. If  $\{X_t : t \geq 0\}$  is locally bounded within  $G$ , then  $X_{t-}$  belongs to  $G$  for all  $t > 0$ , and the process  $f(X_t)$  can be decomposed as  $f(X_t) = M_t + V_t$  where  $M_t$  is a martingale and  $V_t$  is a bounded variation process such that*

$$M_t = f(X_0) + \operatorname{tr} \left( \int_0^t Df(X_{s-}) B_{\mathcal{A}}(ds) \right) + \int_{(0,t] \times \mathbb{H}_d^0} [f(X_{s-} + y) - f(X_{s-})] \tilde{J}_X(ds, dy)$$

and

$$\begin{aligned} V_t = \frac{1}{2} \int_0^t \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} f(X_s) ds + \operatorname{tr} \left( \int_0^t \Psi Df(X_s) ds \right) \\ + \int_{(0,t] \times \mathbb{H}_d^0} [f(X_{s-} + y) - f(X_{s-}) - \operatorname{tr}(Df(X_{s-})y) 1_{\{\|y\| \leq 1\}}] ds \nu(dy). \end{aligned}$$

**Remark 5** *If a Hermitian Lévy process  $X$  is invariant under unitary conjugation with triplet  $(\sigma^2 \mathbf{I}_{d^2}, \nu, \psi \mathbf{I}_d)$ , where  $\nu$  has the polar decomposition  $(\pi, \rho)$ , and  $\pi$  is the uniform measure on  $\mathbb{S}_d$ , then the Itô formula for  $X$  in Proposition 4 becomes in*

$$\begin{aligned} M_t = f(X_0) + \operatorname{tr} \left( \int_0^t Df(X_{s-}) B_{\mathcal{A}}(ds) \right) \\ + \int_{(0,t] \times \mathbb{S}_d \times (0,\infty)} [f(X_{s-} + ru) - f(X_{s-})] (J_X(ds, du, dr) - ds \pi(du) \rho(dr)) \end{aligned}$$

and

$$\begin{aligned} V_t = \frac{\sigma^2}{2} \int_0^t \sum_{(i,j)(k,h)} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} f(X_s) ds + \psi \operatorname{tr} \left( \int_0^t Df(X_s) ds \right) \\ + \int_{(0,t] \times \mathbb{S}_d \times (0,\infty)} [f(X_{s-} + ru) - f(X_{s-}) - r \operatorname{tr}(Df(X_{s-})u) 1_{\{r \leq 1\}}] ds \pi(du) \rho(dr). \end{aligned}$$

### 3 Simultaneous jumps of a Hermitian Lévy process and its eigenvalues

Let  $\{X(t) : t \geq 0\}$  be a  $d \times d$  Hermitian Lévy process and let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t))$  be the vector of eigenvalues of  $X(t)$  where  $\lambda_1(t) \geq \lambda_2(t) \geq \dots \geq \lambda_d(t)$  for each  $t \geq 0$ . If  $\lambda$  jumps at time  $t$ , it is clear that  $\Delta X(t) \neq 0$ , since the  $d$ -tuple of ordered eigenvalues is a continuous function on the space of Hermitian matrices. This follows from the Hoffman-Wielandt inequality (see [1, Rem. 2.1.20]) where for each  $t \geq 0$

$$\|\Delta \lambda(t)\|_2 \leq \|\Delta X(t)\|, \quad (13)$$

where  $\|\cdot\|_2$  denotes the Euclidean norm in  $\mathbb{R}^d$ .

A natural question is whether the jump of the process  $X$  at time  $t$  implies the jump of the eigenvalues at that time. Our goal in this section is to provide conditions for the simultaneous jumps of the eigenvalue process  $\lambda$  whenever  $X$  jumps.

We analyze the non zero jumps of  $X$  in two situations, when  $X(t)$  and  $X(t-)$  commute and when they do not commute. In the commutative case we find that exactly  $k$  eigenvalues jump when  $X$  has a jump of rank  $k$  at time  $t$ , and that all the eigenvalues would jump simultaneously if and only if the jump of  $X$  at time  $t$  is of rank  $d$ . In the noncommutative case, we prove that all the eigenvalues jump simultaneously if the jump of  $X$  at time  $t$  is of rank one.

We present below natural examples of both, commutative and noncommutative situations.

**Example 6** Let  $\{(\lambda_1(t), \dots, \lambda_d(t)) : t \geq 0\}$  be a  $d$ -dimensional Lévy process where  $\{\lambda_i(t) : t \geq 0\}$ ,  $i = 1, \dots, d$ , are independent one-dimensional Lévy processes. For an arbitrary non random unitary  $d \times d$  matrix  $U$ , define the  $d \times d$  Hermitian Lévy process

$$X(t) = U \text{diag}(\lambda_1(t), \dots, \lambda_d(t)) U^*, \quad t \geq 0.$$

Since independent Lévy processes cannot jump at the same time almost surely,  $X$  is an Hermitian Lévy process whose jumps are of rank one almost surely and we observe that in that case  $X(t)$  and  $X(t-)$  are simultaneously diagonalizable.

**Example 7** Let us consider a compound Poisson  $d \times d$  matrix process

$$X(t) = \sum_{j=1}^{N(t)} u_j u_j^*, \quad t \geq 0,$$

where  $u_j, j \geq 1$  are i.i.d. random vectors uniformly distributed on the unit sphere of  $\mathbb{C}^d$  and  $N(t), t \geq 0$ , is a Poisson process independent of  $u_j, j \geq 1$ . Since  $\Delta X(t) = u_{N(t)} u_{N(t)}^* 1_{\{\Delta X(t) \neq 0\}}$  the process  $X$  has jumps of rank one almost surely and  $X(t) = X(t-) + u_{N(t)} u_{N(t)}^*$  whenever there is a jump at time  $t$ . Hence we observe that  $X(t)$  and  $X(t-)$  do not commute since  $u_{N(t)} u_{N(t)}^*$  and  $X(t-) = u_1 u_1^* + \dots + u_{N(t)-1} u_{N(t)-1}^*$  do not commute.

**Example 8** Let us consider  $\{X(t) : t \geq 0\}$  a Hermitian Lévy process in  $\mathbb{H}_d^+$  the set of nonnegative definite  $d \times d$  Hermitian matrices. Assume that  $X$  has jumps of rank one. Then by Theorem 4.1 in [23]  $X$  is the quadratic variation of a  $\mathbb{C}^d$ -valued Lévy process  $\{Y(t) : t \geq 0\}$ , that is

$$X(t) = [Y](t), \quad t \geq 0.$$

For each  $t > 0$ ,  $X(t) = \sum_{0 < u \leq t} \Delta Y_u (\Delta Y_u)^*$  and  $X(t-) = \lim_{s \nearrow t} \sum_{0 < r \leq s} \Delta Y_r (\Delta Y_r)^*$  do not commute in general. In particular,  $X(t)$  and  $X(t-)$  commute if for each  $u, r \in (0, t]$  the term  $\Delta Y_u (\Delta Y_u)^* \Delta Y_r (\Delta Y_r)^*$  is Hermitian.



**Example 9** Let  $\{Z(t) : t \geq 0\}$  be a Hermitian Lévy process in  $\mathbb{H}_d$  of bounded variation with jumps of rank one. By Theorem 4.2 in [23]  $Z$  is the difference of the quadratic variations of two  $\mathbb{C}^d$ -valued Lévy processes  $\{X(t) : t \geq 0\}$  and  $\{Y(t) : t \geq 0\}$ ,

$$Z(t) = [X](t) - [Y](t), \quad t \geq 0,$$

where  $[X](t), t \geq 0$  and  $[Y](t), t \geq 0$  are independent Lévy processes in  $\mathbb{H}_d^+$ . It follows from Example 8 that  $Z(t) = [X](t) - [Y](t)$  and  $Z(t-) = [X](t-) - [Y](t-)$  do not commute in general. In particular,  $Z(t)$  and  $Z(t-)$  commute if for each  $u, r \in (0, t]$  the terms  $\Delta X_u (\Delta X_u)^* \Delta X_r (X_r)^*$ ,  $\Delta Y_u (\Delta Y_u)^* \Delta Y_r (\Delta Y_r)^*$  and  $\Delta X_u (\Delta X_u)^* \Delta Y_r (\Delta Y_r)^*$  are Hermitian.

### 3.1 Commutative case

Assume that  $X(t)$  and  $X(t-)$  commute. Then there exists a random unitary matrix  $U = U_t$  such that  $X(t) = U \text{diag}(\lambda_1(t), \dots, \lambda_d(t)) U^*$  and  $X(t-) = U \text{diag}(\lambda_1(t-), \dots, \lambda_d(t-)) U^*$ . Observe that the eigenvalues of  $\Delta X(t) = X(t) - X(t-)$  are the difference between the eigenvalues of  $X(t)$  and the eigenvalues of  $X(t-)$ .

If  $\Delta X(t)$  is of rank one, there is exactly one eigenvalue  $\lambda_i$  such that  $\Delta X(t) = U \text{diag}(0, \dots, \Delta \lambda_i(t), \dots, 0) U^*$  and all the other eigenvalues do not jump at that time  $t$ . Similarly if  $\Delta X(t)$  is of rank  $k$  where  $1 \leq k \leq d$  there are exactly  $k$  eigenvalues that jump at time  $t$  and all other  $d - k$  eigenvalues remain without jumping at  $t$ . Observe that the only case when all eigenvalues jump simultaneously at  $t$  is when the jump  $\Delta X(t) \neq 0$  is of rank  $d$ .

### 3.2 Noncommutative case

When  $X(t)$  and  $X(t-)$  do not commute the situation is completely different from the commutative case, since the eigenvalues of  $X(t) - X(t-)$  are not any more the difference between the eigenvalues of  $X(t)$  and the eigenvalues of  $X(t-)$ . In this case we prove that if the jump  $\Delta X(t)$  is of rank one, all eigenvalues jump simultaneously at this time  $t$ , as we will see in Theorem 12.

We first consider some facts about the eigenvalues of rank one matrix perturbations. Let  $A$  be a  $d \times d$  complex matrix. A rank one perturbation of  $A$  is a  $d \times d$  complex matrix  $A + ruv^\top$  where  $u, v \in \mathbb{C}^d$  and  $r \in \mathbb{C}$ . If  $S$  is any set of polynomials in  $\mathbb{C}[x_1, \dots, x_d]$ , we define  $V(S) = \{u \in \mathbb{C}^d : f(u) = 0 \text{ for all } f \in S\}$ . If  $S$  is a finite set of polynomials  $f_1, \dots, f_n$  we write  $V(f_1, \dots, f_n)$  for  $V(S)$ . Recall that a subset  $W$  of  $\mathbb{C}^d$  is an algebraic set if  $W = V(f_1, \dots, f_n)$  for some  $f_1, \dots, f_n$ . A non empty subset  $Q \subset \mathbb{C}^d$  is generic if the complement  $\mathbb{C}^d \setminus Q$  is contained in an algebraic set  $W$  which is not  $\mathbb{C}^d$ . In such a case,  $\mathbb{C}^d \setminus Q$  is nowhere dense and of  $2d$  dimensional Lebesgue measure zero.

When a Hermitian Lévy process  $X$  has simple spectrum for each  $t > 0$ , then  $X_t$  and  $X_{t-}$  have no common eigenvalues when the jump at time  $t$  is of rank one.

**Proposition 10** Let  $\{X_t : t \geq 0\}$  be an absolutely continuous  $d \times d$  Hermitian Lévy process. Let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t))$  be the vector of eigenvalues of  $X_t$  for each  $t \geq 0$ . If  $X$  has a rank one jump at time  $t > 0$ , then

$$\{\lambda_1(X_t), \dots, \lambda_d(X_t)\} \cap \{\lambda_1(X_{t-}), \dots, \lambda_d(X_{t-})\} = \emptyset \text{ a.s.}$$

**Proof.** Observe that  $Q = \{z \in \mathbb{C}^{2d} : \text{tr}(zz^*) \neq 0\}$  is a generic set since  $\{z \in \mathbb{C}^{2d} : \text{tr}(zz^*) = 0\} = \{0\}$  is an algebraic set which is equal to  $V(f_1, \dots, f_{2d})$  where  $f_i(z) = z_i$  for any  $z = (z_1, \dots, z_{2d})^\top \in \mathbb{C}^d$  and  $i = 1, \dots, 2d$ .

Since  $X$  has a jump of rank one at time  $t$ ,  $\Delta X_t = ruu^*$  with  $r = r_t \in \mathbb{R} \setminus \{0\}$  and  $u = u_t \in \mathbb{C}^d \setminus \{0\}$ . That is, we have the rank one perturbation  $X_{t-} = X_t + ruu^*$ . Since  $X_t$  is absolutely continuous and Hermitian, the eigenvalues  $\lambda_1(X_t), \dots, \lambda_d(X_t)$  do not collide almost surely, hence the degree of the minimal polynomial of  $X_t$  is  $d$ . By Proposition 2.2 (iii) of [31] applied to  $z = (u, \bar{u}) \in Q$ , there are exactly  $d$  eigenvalues of  $X_{t-}$  that are not eigenvalues of  $X_t$ . ■

**Remark 11** Proposition 2.2 in [31] also gives insight into the behavior of the non-Hermitian rank one jumps of a general complex matrix Lévy process for which  $X_t$  no longer has a simple spectrum. In this case,  $\Delta X_t = ruv^\top$  with  $r \in \mathbb{C} \setminus \{0\}$  and  $u, v \in \mathbb{C}^d \setminus \{0\}$ , the number  $m_t \leq d$  of distinct eigenvalues of  $X_t$  might be random and there are exactly  $m_t$  eigenvalues of  $X_{t-}$  that are not eigenvalues of  $X_t$ .

We now present the main result of this paper, giving conditions for the simultaneous jumps of a Hermitian Lévy process and its eigenvalue process. Moreover, if one eigenvalue component of the eigenvalue process jumps, then all the other eigenvalue components jump.

**Theorem 12** Let  $\{X(t) : t \geq 0\}$  be an absolutely continuous  $d \times d$  Hermitian Lévy process and let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t))$  be the vector of eigenvalues of  $X(t)$  where  $\lambda_1(t) \geq \lambda_2(t) \geq \dots \geq \lambda_d(t)$  for each  $t \geq 0$ .

If  $X$  has a jump of rank one at  $t > 0$  and  $X(t)$  and  $X(t-)$  do not commute then  $\Delta\lambda(t) \neq 0$ . Moreover  $\Delta\lambda_i(t) \neq 0$  for some  $i$  if and only if  $\Delta\lambda_j(t) \neq 0$  for all  $j \neq i$ .

**Proof.** If the Lévy process  $X$  jumps at time  $t$ , then, by Proposition 10,  $\lambda_i(X_t) - \lambda_j(X_{t-}) \neq 0$  for all  $i, j = 1, 2, \dots, d$  and therefore  $\Delta\lambda(t) \neq 0$ .

Moreover, if  $\lambda$  jumps at time  $t$ , it is clear from (13) that  $\Delta X(t) \neq 0$ , and then by the first part of the proof all the eigenvalues jump, that is  $\Delta\lambda_j(t) \neq 0$  for all  $j \neq i$ . ■

**Remark 13** a) We finally observe that Proposition 10 and Theorem 12 are true also for a Hermitian semimartingale  $X$  with simple spectrum for each  $t > 0$ . This is guaranteed whenever  $X(t)$  has an absolutely continuous distribution for each  $t > 0$ .

b) However, we have written the results of this section in the framework of Hermitian Lévy processes since the subject is one of Ole's interests.

## 4 The dynamics of the eigenvalues of Hermitian Lévy processes

### 4.1 Eigenvalue derivatives of Hermitian Lévy processes

Consider a Hermitian matrix  $X$  with decomposition  $X = UDU^*$  given by (6) where  $U = (u_{ij})$  is a unitary matrix, with  $u_{ii} > 0$  for all  $i$ ,  $u_{ij} \neq 0$  for  $i, j$ , and all minors of  $U$  having non-zero determinants and  $D$  is a diagonal matrix with entries  $\lambda_i = D_{ii}$  such that  $\lambda_1 > \lambda_2 > \dots > \lambda_d$ .

Now if we assume that  $X = (x_{ij} + iy_{ij})_{1 \leq i, j \leq d}$  is a smooth function of a real parameter  $\theta$ , the Hadamard variation formulae hold: for each  $m = 1, 2, \dots, d$ ,

$$\begin{aligned} \partial_\theta \lambda_m &= (U^* \partial_\theta X U)_{mm}, \\ \partial_\theta^2 \lambda_m &= (U^* \partial_\theta^2 X U)_{mm} + 2 \sum_{j \neq m} \frac{|(U^* \partial_\theta X U)_{mj}|^2}{\lambda_m - \lambda_j}. \end{aligned}$$

In particular, for  $\theta = x_{kh}$  and  $y_{kh}$  with  $k \leq h$ , we observe that

$$\begin{aligned} \frac{\partial \lambda_m}{\partial x_{kh}} &= 2 \operatorname{Re}(\bar{u}_{km} u_{hm}) 1_{\{k \neq h\}} + 2 |u_{km}|^2 1_{\{k=h\}}, \\ \frac{\partial \lambda_m}{\partial y_{kh}} &= 2 \operatorname{Im}(\bar{u}_{km} u_{hm}) 1_{\{k \neq h\}}, \\ \frac{\partial^2 \lambda_m}{\partial x_{kh}^2} &= 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{hj} + \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k \neq h\}} + 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k=h\}}, \end{aligned}$$

$$\frac{\partial^2 \lambda_m}{\partial y_{kh}^2} = 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{hj} - \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k \neq h\}}.$$

We point out that the spectral dynamics of Hermitian matrices is studied in [33], giving a Hadamard variation formula of any order for each eigenvalue in terms of a gradient. It is noted there that an eigenvalue depends smoothly on  $X$  whenever that eigenvalue is simple (even if other eigenvalues are not).

Using the former derivatives of the eigenvalues and the smooth function given in (8) we have the following result for Hermitian Lévy processes with simple spectrum for each time  $t > 0$ .

**Lemma 14** *Let  $\{X_t : t \geq 0\}$  be a  $d \times d$  Hermitian Lévy process with absolutely continuous distribution for each  $t > 0$ . Let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t))$  with  $\lambda_1(t) > \dots > \lambda_d(t)$  the eigenvalues of  $X_t$  for each  $t \geq 0$ . For each  $m = 1, 2, \dots, d$ , there exists  $\Phi_m : \mathbb{R}^{d^2} \rightarrow \mathbb{R}$  which is  $C^\infty$  in an open subset  $\tilde{G} \subset \mathbb{R}^{d^2}$ , with  $\text{Leb}(\mathbb{R}^{d^2} \setminus \tilde{G}) = 0$ , such that for each  $t > 0$ ,  $\lambda_m(t) = \Phi_m(X_t)$  and for  $k \leq h$*

$$\frac{\partial \Phi_m}{\partial x_{kh}} = 2 \text{Re}(\bar{u}_{km} u_{hm}) 1_{\{k \neq h\}} + 2 |u_{km}|^2 1_{\{k=h\}}, \quad (14)$$

$$\frac{\partial \Phi_m}{\partial y_{kh}} = 2 \text{Im}(\bar{u}_{km} u_{hm}) 1_{\{k \neq h\}}, \quad (15)$$

and

$$\frac{\partial^2 \Phi_m}{\partial x_{kh}^2} = 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{hj} + \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k \neq h\}} + 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k=h\}}, \quad (16)$$

$$\frac{\partial^2 \Phi_m}{\partial y_{kh}^2} = 2 \sum_{j \neq m} \frac{|\bar{u}_{km} u_{hj} - \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} 1_{\{k \neq h\}}, \quad (17)$$

where for simplicity, the dependence on  $t$  is omitted in  $\lambda_m = \lambda_m(t)$  and  $u_{kl} = u_{kl}(t)$ , which are random.

## 4.2 The SDE for the process of eigenvalues

Using Lemma 14 and Proposition 3, we can describe the dynamics of the process of eigenvalues of a Hermitian Lévy process.

**Theorem 15** *Let  $\{X_t : t \geq 0\}$  be a Hermitian Lévy process with triplet  $(\mathcal{A}, \nu, \Psi)$  and locally bounded within the open set  $G$  in  $\mathbb{H}_d$  identified with the open set  $\tilde{G}$  in  $\mathbb{R}^{d^2}$  of Lemma 14. Assume that  $X_t$  has an absolutely continuous distribution for each  $t > 0$ . Let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_d(t))$  be the eigenvalues of  $X_t$  for each  $t \geq 0$ .*

*Then, for each  $m = 1, \dots, d$  the process  $\lambda_m(X_t)$  is a semimartingale and*

$$\begin{aligned} \lambda_m(X_t) &= \lambda_m(X_0) + \text{tr} \left( \int_0^t D\lambda_m(X_{s-}) dX_s \right) + \frac{1}{2} \int_0^t \sum_{(i,j) \in (k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} \partial z_{kh}} \lambda_m(X_s) ds \\ &\quad + \int_{(0,t] \times \mathbb{H}_d^0} [\lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) - \text{tr}(D\lambda_m(X_{s-})y)] J_X(ds, dy), \end{aligned} \quad (18)$$

where a)

$$\begin{aligned} D\lambda_m(X_{s-}) &= \left( \frac{\partial \Phi_m(s-)}{\partial x_{kh}} + i \frac{\partial \Phi_m(s-)}{\partial y_{kh}} 1_{\{k \neq h\}} \right)_{1 \leq k \leq h \leq d}, \\ \frac{\partial \Phi_m(s-)}{\partial x_{kh}} &= 2 \text{Re}(\bar{u}_{km}(s-) u_{hm}(s-)) 1_{\{k \neq h\}} + 2 |u_{km}(s-)|^2 1_{\{k=h\}}, \end{aligned} \quad (19)$$

$$\frac{\partial \Phi_m(s-)}{\partial y_{kh}} = 2 \operatorname{Im} (\bar{u}_{km}(s-) u_{hm}(s-)) 1_{\{k \neq h\}}. \quad (20)$$

b)

$$\begin{aligned} \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} z_{kh}} \lambda_m(X_s) &= 2 \sum_{k=1}^d A_{kk,kk} \sum_{j \neq m} \frac{|u_{km}(s) u_{kj}(s)|^2}{\lambda_m(s) - \lambda_j(s)} \\ &+ 4 \sum_{1 \leq k < h \leq d} A_{kh,kh} \sum_{j \neq m} \frac{|u_{km}(s) u_{hj}(s)|^2 + |u_{hm}(s) u_{kj}(s)|^2}{\lambda_m(s) - \lambda_j(s)}. \end{aligned} \quad (21)$$

**Proof.** By Lemma 14, for each  $m = 1, 2, \dots, d$  there exists  $\Phi_m : \mathbb{R}^{d^2} \rightarrow \mathbb{R}$  which is  $C^\infty$  in an open subset  $\tilde{G}$  of  $\mathbb{R}^{d^2}$  such that  $\lambda_m(t) = \Phi_m(X_t)$  for each  $t > 0$ , and the  $\Phi_m$  satisfy (14)–(17).

Now, taking  $f = \lambda_m$  in Proposition 3, where the open set  $G$  is identified with the open subset  $\tilde{G}$ , we have that  $\lambda_m(X_t)$  is a semimartingale where formula (12) becomes (18). The derivative matrix  $D\lambda_m(X_{s-})$  in (18), see (9), is given by  $D\lambda_m(X_{s-}) = \left( \frac{\partial \Phi_m(s-)}{\partial x_{kh}} + i \frac{\partial \Phi_m(s-)}{\partial y_{kh}} 1_{\{k \neq h\}} \right)_{1 \leq k \leq h \leq d}$ , where the expressions of  $\frac{\partial \Phi_m(s-)}{\partial x_{kh}}$  and  $\frac{\partial \Phi_m(s-)}{\partial y_{kh}}$  in (14) and (15) turn out to be in (19) and (20).

The second derivative term in (18) is given by  $\sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} z_{kh}} \lambda_m(X_s) = \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} z_{kh}} \Phi_m(s)$ . It can be seen that  $\frac{\partial^2}{\partial x_{ii} x_{jj}} \Phi_m(s) = 0$  for  $i \neq j$ ,  $\frac{\partial^2}{\partial x_{ij} x_{kh}} \Phi_m(s) = 0$  for  $i < j, k < h$  with  $ij \neq kh$ ,  $\frac{\partial^2}{\partial x_{ij} y_{kh}} \Phi_m(s) = 0$  for  $i \leq j, k < h$ , and  $\frac{\partial^2}{\partial y_{ij} y_{kh}} \Phi_m(s) = 0$  for  $i < j, k < h$  with  $ij \neq kh$ .

Using the previous computations, the operator (11) applied to  $\Phi_m(s)$  gives

$$\begin{aligned} \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} z_{kh}} \Phi_m(s) &= \sum_{k=1}^d A_{kk,kk} \frac{\partial^2}{\partial x_{kk}^2} \Phi_m(X_s) \\ &+ \sum_{1 \leq k < h \leq d} A_{kh,kh} \left( \frac{\partial^2}{\partial x_{kh}^2} + \frac{\partial^2}{\partial y_{kh}^2} \right) \Phi_m(X_s), \end{aligned} \quad (22)$$

and now from (16) and (17) we get

$$\sum_{k=1}^d A_{kk,kk} \frac{\partial^2}{\partial x_{kk}^2} \Phi_m(X_s) = 2 \sum_{k=1}^d A_{kk,kk} \sum_{j \neq m} \frac{|\bar{u}_{km} u_{kj}|^2}{\lambda_m - \lambda_j}, \quad (23)$$

and

$$\sum_{1 \leq k < h \leq d} A_{kh,kh} \left( \frac{\partial^2}{\partial x_{kh}^2} + \frac{\partial^2}{\partial y_{kh}^2} \right) \Phi_m(X_s) \quad (24)$$

$$\begin{aligned} &= 2 \sum_{1 \leq k < h \leq d} A_{kh,kh} \sum_{j \neq m} \left\{ \frac{|\bar{u}_{km} u_{hj} + \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} + \frac{|\bar{u}_{km} u_{hj} - \bar{u}_{hm} u_{kj}|^2}{\lambda_m - \lambda_j} \right\} \\ &= 2 \sum_{1 \leq k < h \leq d} A_{kh,kh} \sum_{j \neq m} 2 \frac{|\bar{u}_{km}(s) u_{hj}(s)|^2 + |\bar{u}_{hm}(s) u_{kj}(s)|^2}{\lambda_m(s) - \lambda_j(s)}. \end{aligned}$$

Finally, (21) follows from (22), (23), (24). ■

**Remark 16** (a) Theorem 15 is the analog of the Dyson–Brownian motion (1) for a Hermitian Lévy process. In this case,  $X(t)$  has a simple spectrum almost surely for each  $t > 0$ . The problem of noncolliding eigenvalues for each  $t > 0$  almost surely will be considered elsewhere.

(b) However, we observe that the repulsion term between each pair of eigenvalues  $1/(\lambda_i(s) - \lambda_j(s))$ ,  $i \neq j$ , appears in (18) and (21) when there is a Gaussian component (i.e.,  $\mathcal{A} \neq 0$ ).

(c) The corresponding expression of Proposition 4 can be obtained for the dynamics of eigenvalues in terms of the local martingale and bounded variation components of the eigenvalues. Let  $\{X_t : t \geq 0\}$  be a Hermitian Lévy process with triplet  $(\mathcal{A}, \nu, \Psi)$  satisfying the same conditions as in Theorem 15. For each  $m = 1, 2, \dots, d$  it holds the decomposition  $\lambda_m(X_t) = M_t^m + V_t^m$  where  $M_t^m$  is a martingale and  $V_t^m$  is a process of bounded variation such that

$$M_t^m = \lambda_m(X_0) + \text{tr} \left( \int_0^t D\lambda_m(X_{s-}) B_{\mathcal{A}}(ds) \right) + \int_{(0,t] \times \mathbb{H}_d^0} [\lambda_m(X_{s-} + y) - \lambda_m(X_{s-})] \tilde{J}_X(ds, dy)$$

and

$$V_t^m = \frac{1}{2} \int_0^t \sum_{(i,j)(k,h)} A_{ij,kh} \frac{\partial^2}{\partial z_{ij} \partial \bar{z}_{kh}} \lambda_m(X_s) ds + \text{tr} \left( \int_0^t \Psi D\lambda_m(X_s) ds \right) + \int_{(0,t] \times \mathbb{H}_d^0} [\lambda_m(X_{s-} + y) - \lambda_m(X_{s-}) - \text{tr}(D\lambda_m(X_{s-})y) 1_{\{\|y\| \leq 1\}}] d\nu(dy).$$

d) Hermitian Lévy processes with rank one jumps and with their distributions invariant under unitary conjugations appear in the context of a connection between classical and free one-dimensional infinitely divisible distributions [13], [20], [22], [23]. Let us consider an example with the triplet  $(\sigma^2 \mathbf{I}_{d^2}, \nu, \psi \mathbf{I}_d)$ . From (c), for each  $m = 1, \dots, d$ , the eigenvalue  $\lambda_m$  has the decomposition  $\lambda_m(t) = M_t^m + V_t^m$ , where

$$M_t^m = \lambda_m(X_0) + \text{tr} \left( \int_0^t D\lambda_m(X_{s-}) B_{\mathcal{A}}(ds) \right) + \int_{(0,t] \times \mathbb{H}_d^1} [\lambda_m(X_{s-} + u) - \lambda_m(X_{s-})] (J_X(ds, du) - d\nu(du))$$

and

$$V_t^m = 2\sigma^2 \int_0^t \sum_{j \neq m} \frac{1}{\lambda_m(X_s) - \lambda_j(X_s)} ds + \psi \text{tr} \left( \int_0^t D\lambda_m(X_s) ds \right) + \int_{(0,t] \times \mathbb{H}_d^1} [\lambda_m(X_{s-} + u) - \lambda_m(X_{s-}) - \text{tr}(D\lambda_m(X_{s-})u) 1_{\{\|u\| \leq 1\}}] d\nu(du).$$

In particular, when the process is symmetric, the Lévy measure is symmetric and  $\psi = 0$ , and hence

$$V_t^m = 2\sigma^2 \int_0^t \sum_{j \neq m} \frac{1}{\lambda_m(X_s) - \lambda_j(X_s)} ds + \int_{(0,t] \times \mathbb{H}_d^1} [\lambda_m(X_{s-} + u) - \lambda_m(X_{s-})] d\nu(du).$$

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## References

- [1] W. Anderson, A. Guionnet and O. Zeitouni (2009): *An Introduction to Random Matrices*. Cambridge University Press.

- [2] D. Applebaum (2007): Lévy processes and stochastic integrals in Banach spaces. *Probab. Math. Statist.* **27**, 75–88.
- [3] O. E. Barndorff-Nielsen, M. Maejima and K. Sato (2004): Some classes of multivariate infinitely divisible distributions admitting stochastic integral representation. *Bernoulli* **12**, 1–33.
- [4] O. E. Barndorff-Nielsen and V. Pérez-Abreu (2008): Matrix subordinators and related Upsilon transformations. *Theory Probab. Appl.* **52**, 1–23.
- [5] O. E. Barndorff-Nielsen, V. Pérez-Abreu and A. Rocha-Arteaga (2006): MatG random matrices. *Stoch. Mod.* **22**, 723–734.
- [6] O. E. Barndorff-Nielsen, V. Pérez-Abreu and S. Thorbjørnsen (2013): Lévy Mixings. *ALEA, Lat. Am. J. Probab. Math. Stat.* **10**, 1013–1062.
- [7] O. E. Barndorff-Nielsen, J. Rosiński and S. Thorbjørnsen (2008): General  $\Upsilon$ -transformations. *ALEA, Lat. Am. J. Probab. Math. Stat.* **4** (2008), 131–165.
- [8] O. E. Barndorff-Nielsen and R. Stelzer (2007): Positive-definite matrix processes of finite variation. *Probab. Math. Statist.* **27**, 3–43.
- [9] O. E. Barndorff-Nielsen and R. Stelzer (2011): The multivariate supOU stochastic volatility model. *Math. Finance.* **23**, 275–296.
- [10] O. E. Barndorff-Nielsen and R. Stelzer (2011): Multivariate supOU processes. *Ann. Appl. Probab.* **21**, 140–182.
- [11] O. E. Barndorff-Nielsen and S. Thorbjørnsen (2004): A connection between free and classical infinite divisibility. *Inf. Dim. Anal. Quantum Probab.* **7**, 573–590.
- [12] O. E. Barndorff-Nielsen and S. Thorbjørnsen (2006): Classical and free infinite divisibility and Lévy processes. In: *Quantum Independent Increment Processes II*. M. Schürmann and U. Franz (eds). *Lecture Notes in Mathematics* **1866**, Springer-Verlag, Berlin.
- [13] F. Benaych-Georges (2005): Classical and free infinitely divisible distributions and random matrices. *Ann. Probab.* **33**, 1134–1170.
- [14] H. Bercovici and V. Pata, with an appendix by P. Biane (1999): Stable laws and domains of attraction in free probability theory. *Ann. Math.* **149**, 1023–1060.
- [15] R. Bhatia (1997): *Matrix Analysis*. *Graduate Texts in Mathematics* **169**, Springer-Verlag, Berlin.
- [16] K. Bichteler (2002): *Stochastic Integration with Jumps*. Cambridge University Press.
- [17] M. F. Bru (1989): Diffusions of perturbed principal component analysis. *J. Multivariate Anal.* **29**, 127–136.
- [18] M. F. Bru (1991): Wishart processes. *J. Theoret. Probab.* **9**, 725–751.
- [19] T. Chan (1992): The Wigner semicircle law and eigenvalues of matrix diffusions. *Probab. Theory Relat. Fields* **93**, 249–272.
- [20] T. Cabanal-Duvillard (2005): A matrix representation of the Bercovici–Pata bijection. *Electron. J. Probab.*, **10**, 632–661.
- [21] R. Cont and P. Tankov (2004): *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, Chichester, UK.

- [22] J. A. Domínguez-Molina and A. Rocha-Arteaga (2012): Random matrix models of stochastic integrals type for free infinitely divisible distributions. *Period. Math. Hung.* **64** (2), 145–160.
- [23] J. A. Domínguez-Molina, V. Pérez-Abreu and A. Rocha-Arteaga (2013): Covariation representation for Hermitian Lévy process ensembles of free infinitely divisible distributions. *Electron. Comm. Probab.* **18** 1–14.
- [24] F. J. Dyson (1962): A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3**, 1191–1198.
- [25] M. Katori and K. Tanemura (2004): Symmetry of matrix stochastic processes and noncolliding diffusion particle systems. *J. Math. Phys.* **45**, 3058–3085.
- [26] M. Katori and K. Tanemura. (2013): Complex Brownian motion representation of the Dyson model. *Elect. Comm. Probab.* **18**, 1–16.
- [27] W. König and N. O Connell (2001): Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Elect. Comm. Probab.* **6**, 107–114.
- [28] D. Nualart and V. Pérez-Abreu (2014): On the eigenvalue process of a matrix fractional Brownian motion. *Stoch. Process. Appl.* **124**, 4266–4282.
- [29] V. Pérez-Abreu and N. Sakuma (2008): Free generalized gamma convolutions. *Elect. Comm. Probab.* **13**, 526–539.
- [30] P. Protter (2004): *Stochastic Integration and Differential Equations. Stoch. Model. Appl. Probab.* **21**, Springer-Verlag, Berlin.
- [31] A. C. M. Ran and M. Wojtylak (2012): Eigenvalues of rank one perturbations of unstructured matrices. *Linear Algebra and its Applications*, **437**, 589–600.
- [32] K. Sato (1999): *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press.
- [33] T. Tao and V. Vu (2011): Random matrices: Universality of local eigenvalue statistics. *Acta Math.*, **206**, 127–204.
- [34] T. Tao. (2012): *Topics in Random Matrix Theory. Graduate Studies in Mathematics* **132**, American Mathematical Society, Providence, RI.